THE PRESSURE OF A RIGID BODY ON PLATES AND MEMBRANES

(DAVLENIE TVERDOGO TELA NA PLASTINY I MEMBRANY)

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The paper investigates the problem of the pressure of a rigid paraboloid of revolution on a plate or a membrane the contour of which consists of portions of straight lines; the plate is assumed to be simply supported. The problem is solved in quadratures and the method of solution is based on the reduction to a type of Riemann boundary-value problems for two functions for which a solution in closed form has been found. The problem of the pressure of a rigid paraboloid on an infinite plate was first solved by Galin [1].

1. **General relations and formulation of the problem.** (1). The deflection of a stiff elastic plate under the action of a single transverse loading satisfies the Kirchhoff equation [2]

$$D\Delta\Delta w = q \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1.1}$$

Here w is the displacement of points in the plate along the normal to its surface, D is the bending stiffness of the plate and q is the transverse loading.

We have the following formulas for the bending moments M_t and M_n [2]

$$M_t = -D\left(\frac{\partial^2 w}{\partial t^2} + \nu\frac{\partial^2 w}{\partial n^2}\right), \qquad M_n = -D\left(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial t^2}\right) \tag{1.2}$$

Here tn is an arbitrary system of Cartesian coordinates an ν is Poisson's ratio:

On a part of the plate where the transverse loading q is zero there exist the fundamental relations [3]

$$\begin{gathered}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 4\operatorname{Re}\Phi(z) \qquad (z = x + iy)\\ \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - 2i\frac{\partial^2 w}{\partial x \partial y} = 2\left[\bar{z}\Phi'(z) + \Psi'(z)\right]\end{gathered} \tag{1.3}$$

where $\Phi(z)$ and $\Psi(z)$ are analytic functions; x, y are Cartesian coordinates.

We give two further formulas for the case when $q = 0$ which will prove of use at a later stage [3]

$$\begin{gathered}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{\partial^2 w}{\partial n^2}+\frac{\partial^2 w}{\partial t^2} \\ \frac{\partial^2 w}{\partial t^2}-\frac{\partial^2 w}{\partial n^2}-2i\,\frac{\partial^2 w}{\partial n \partial t}=e^{2i\alpha}\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}-2i\,\frac{\partial^2 w}{\partial x \partial y}\right)\end{gathered} \tag{1.4}$$

where α is the angle between the axes x and t.

An absolutely rigid paraboloid of revolution, the equation of the surface of which can be written in the form

$$z = Ax^2 + Ay^2 \tag{1.5}$$

bears on a plate with a force P directed along the axis of symmetry. Over the contact area the deflection of the plate will be given by

$$w = \delta - Ax^2 - Ay^2 \tag{1.6}$$

where δ is the displacement of the die. The thickness of the plate is assumed to be small compared with the dimensions of the contact area. On the contour L of this area, which is to be determined, the following conditions of continuity of all second derivatives [1] of the function w must be satisfied:

$$\frac{\partial^2 w}{\partial x^2} = -2A, \qquad \frac{\partial^2 w}{\partial y^2} = -2A, \qquad \frac{\partial^2 w}{\partial x \partial y} = 0 \tag{1.7}$$

These conditions are a consequence of the continuity of the displacement w and also of its first and second derivatives with respect to the normal, $\partial w/\partial n$ and $\partial^2 w/\partial n^2$. By virtue of (1.1) and (1.6) the pressure on the contact area is zero, so that the force P is resisted by shears acting on the boundary of the contact area (*).

ii) When the bending stiffness of the plate can be ignored and the stresses on the middle surface caused by bending are small compared with the initial stresses σ_x and σ_y, the deflection w of the plate satisfies the equation

$$\sigma_x \frac{\partial^2 w}{\partial x^2} + \sigma_y \frac{\partial^2 w}{\partial y^2} = -\frac{q}{h} \tag{1.8}$$

Here σ_x and σ_y are constant principal stresses and h is the thickness of the membrane. Note that Equation (1.8) is valid also for nonelastic membranes.

For the case when the transverse loading q is zero we can use the fundamental relations

$$\begin{gathered} w = \operatorname{Re} f(z), \qquad \frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y} = \operatorname{Re} f'(z) + i\varkappa \operatorname{Im} f'(z) \\ z = x + i\varkappa y, \qquad \varkappa^2 = \sigma_x/\sigma_y \end{gathered} \tag{1.9}$$

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*) If the effect of shear forces is taken into account the pressure distribution under the die is found to be nonzero, as has been proved in [4].

An absolutely rigid paraboloid of revolution, the equation of the surface of which may be written the form (1.5) is considered to bear on the membrane. On the basis of (1.8) and (1.6) the pressure over the area of contact is constant and equal to

$$q = 2Ah\,(\sigma_x + \sigma_y) \tag{1.10}$$

On the contour L of the contact area, which is to be determined, the condition of continuity of both first derivatives of the function w must be satisfied

$$\frac{\partial w}{\partial x} = -2Ax, \qquad \frac{\partial w}{\partial y} = -2Ay \tag{1.11}$$

2. **Auxiliary boundary-value problem.** Suppose it is required to determine two piecewise analytic functions $\varphi_1(z)$ and $\varphi_2(z)$ of a complex variable z with a line of discontinuities $L + M$, the boundary values of which on the contour $L + M$ satisfy the conditions (2.1)

$$\varphi_1^+(t) = \alpha_1(t)\,\varphi_2^-(t) + f_1(t), \qquad \varphi_2^+(t) = \beta_1(t)\,\varphi_1^-(t) + f_2(t) \qquad (t \in L)$$

$$\varphi_1^+(t) = g_1(t)\,\varphi_1^-(t) + f_3(t), \qquad \varphi_2^+(t) = g_2(t)\,\varphi_2^-(t) + f_4(t) \qquad (t \in M)$$

Here $\alpha_1(t), \beta_1(t), g_1(t), g_2(t), f_1(t), \ldots, f_4(t)$ are piecewise continuous functions satisfying Hölder's condition over the intervals of continuity; $L + M$ is a simple smooth contour.

Thus on the part M of the contour we have a linear boundary condition of the Riemann problem for each function $\varphi_1(z)$ and $\varphi_2(z)$ taken separately [5 and 6] and on the remaining portion L of the contour we have a linear boundary-value problem of Riemann for the two functions of the type considered in [7].

We introduce new piecewise analytic functions $\psi_1(z)$ and $\psi_2(z)$ by means of Formulas

$$\psi_1(z) = \frac{\varphi_1(z)}{X_1(z)} - \frac{1}{2\pi i}\int_M \frac{f_3(t)\,dt}{X_1^+(t)\,(t-z)}$$

$$\psi_2(z) = \frac{\varphi_2(z)}{X_2(z)} - \frac{1}{2\pi i}\int_M \frac{f_4(t)\,dt}{X_2^+(t)\,(t-z)} \tag{2.2}$$

Here $X_i(z)$ is the canonical solution of the Riemann boundary-value problem [5 and 6]

$$\varphi_i^+(t) = g_i(t)\,\varphi_i^-(t) \qquad (t \in M)\ (i = 1, 2) \tag{2.3}$$

(where it is assumed that $\varphi_i(z)$ are analytic everywhere outside the contour M).

The functions $\psi_1(z)$ and $\psi_2(z)$ determined in this way, as can be easily verified, undergo a discontinuity only on the line L and are continuous on the contour M

$$\psi_1^+ = \psi_1^-, \qquad \psi_2^+ = \psi_2^- \qquad (t \in M)$$

Substituting the functions $\varphi_1(z)$ and $\varphi_2(z)$ according to (2.2) into the first two boundary conditions of (2.1), we obtain the following boundary-value problem for the functions $\psi_1(z)$ and $\psi_2(z)$:

$$\psi_1^+(t) = \alpha(t)\,\psi_2^-(t) + f_{10}(t), \qquad \psi_2^+(t) = \beta(t)\,\psi_1^-(t) + f_{20}(t) \qquad (t \in L) \tag{2.4}$$

Here

$$\alpha(t) = \frac{X_2(t)}{X_1(t)}\alpha_1(t), \quad \beta(t) = \frac{X_1(t)}{X_2(t)}\beta_1(t)$$

$$f_{10}(t) = \frac{1}{X_1(t)}\left\{f_1(t) + \alpha_1(t)\frac{X_2(t)}{2\pi i}\int_M \frac{f_4(\tau)\,d\tau}{X_2^+(\tau)(\tau - t)} - \frac{X_1(t)}{2\pi i}\int_M \frac{f_3(\tau)\,d\tau}{X_1^+(\tau)(\tau - t)}\right\}$$

$$f_{20}(t) = \frac{1}{X_2(t)}\left\{f_2(t) + \beta_1(t)\frac{X_1(t)}{2\pi i}\int_M \frac{f_3(\tau)\,d\tau}{X_1^+(\tau)(\tau - t)} - \frac{X_2(t)}{2\pi i}\int_M \frac{f_4(\tau)\,d\tau}{X_2^+(\tau)(\tau - t)}\right\}$$

The boundary-value problem (2.4) has been solved in closed form in [7].

Since an error occurred in Expressions (1.9) in [7] the correct version of these formulas is quoted here

$$\psi_j(z) = X_j(z)\left\{\sum_{k=1}^{\nu}\frac{a_{jk}}{z - z_{jk}} + P_\lambda(z) + \frac{1}{4\pi i}\int_L\left[\frac{f_{10}(t)}{X_1^+(t)} + \frac{f_{20}(t)}{X_2^+(t)}\right]\frac{dt}{t - z} + \right.$$

$$+ (-1)^{j-1}\frac{R_2(z)}{R_1(z)}\left[P_\mu(z) + \frac{1}{4\pi i}\int_L \frac{R_1^+(t)}{R_2^+(t)}\left\langle\frac{f_{10}(t)}{X_1^+(t)} - \frac{f_{20}(t)}{X_2^+(t)} + \right.\right.$$

$$\left.\left.\left. + 2\sum_{k=1}^{\nu}\left(\frac{a_{2k}}{t - d_k} - \frac{a_{1k}}{t - c_k}\right)\right\rangle\frac{dt}{t - z}\right]\right\} \qquad (2.5)$$

Here $X_1(z)$, $X_2(z)$ is the canonical solution [7] of the boundary-value problem (2.4); $z_{1k} = c_k$, $z_{2k} = d_k$ are certain complex constants [7]; the constants a_{jk} are given by $a_{jk} = \lim\,(z - z_{jk})\,\psi_j(z)\,X_j^{-1}(z)$ as $z \to z_{jk}$; $P_n(z)$ is a polynomial of degree n. The numbers λ and μ are determined as follows:

1) $\nu - \varkappa - l + n > 0$, $\mu = 0$, $n - l - 1 > 0$, $\lambda = n - l - 1$

2) $\nu - \varkappa - l + n > 0$, $\mu = 0$, $n - l - 1 < 0$, $\nu - \varkappa > 0$, $\lambda = 0$

3) $\nu - \varkappa - l + n > 0$, $\mu = 0$, $n - l - 1 < 0$, $\nu - \varkappa < 0$, $\lambda = \varkappa - \nu$

4) $\nu - \varkappa - l + n < 0$, $\mu = -\nu + \varkappa + l - n$, $\varkappa - \nu > 0$, $\lambda = \varkappa - \nu$

5) $\nu - \varkappa - l + n < 0$, $\mu = -\nu + \varkappa + l - n$, $\nu - \varkappa > 0$, $\lambda = 0$

$\varkappa = \varkappa_1 + \varkappa_2 + \ldots + \varkappa_n$; in general $\nu = [{}^1/_2 n]$. $R_1(z)$ and $R_2(z)$ denote the following functions:

$$R_1(z) = \prod_{i=1}^{l}(z - g_i)^{1/2}, \qquad R_2(z) = \prod_{i=l+1}^{2n}(z - g_i)^{1/2}$$

At the ends of the lines L_k $(L_1 + \ldots + L_n = L)$ $z = g_i$ $(i = 1, 2, \ldots, l)$ the functions $X_1(z)$ and $X_2(z)$ are of the order $O\,[(z - g_i)^\alpha]$, where $1 > \operatorname{Re}\alpha \geqslant {}^1/_2$.

In the solution of boundary-value problem (2.1) we can proceed by a different (and perhaps a more convenient) method. We can first construct a canonical solution to (2.1) defined as a solution to the homogeneous problem of (2.1), the class of which coincides with the given class of functions $\varphi_1(x)$ and $\varphi_2(x)$ at the ends of the curves and at the points of discontinuity of the coefficients α_1, β_1, g_1 and g_2 ; in so doing it is necessary in general to introduce certain additional conditions analogous to the condition (1) in [7]. These conditions are not difficult to find after the homogeneous boundary-value problem (2.1) has been solved by the method outlined above. When the canonical solution has been found, the boundary-value problem (2.1) can easily be solved.

3. The pressure of an absolutely rigid paraboloid of revolution on a plate and on a membrane of polygonal shape. (1) Consider an elastic simply connected polygonal plate freely supported at its boundary and subjected to the pressure of an absolutely rigid paraboloid of revolution. On the freely supported boundary M the conditions are that $w = 0$ and $M_n = 0$, or, from (1.2)

$$\frac{\partial^2 w}{\partial t^2} = 0, \qquad \frac{\partial^2 w}{\partial n^2} = 0 \tag{3.1}$$

Here n and t are the directions of the normal and the tangent to the contour of the plate.

Using Formulas (1.3), (1.4) and (1.7) we can write the boundary conditions (3.1) in the form

$$\begin{gathered}\operatorname{Re}\Phi(z) = -A, \qquad \bar{z}\Phi'(z) + \Psi(z) = 0 \quad \text{on } L \\ \operatorname{Re}\Phi(z) = 0, \quad \operatorname{Re}\{e^{2i\alpha_j}[\bar{z}\Phi'(z) + \Psi(z)]\} = 0 \quad \text{on } M\end{gathered} \tag{3.2}$$

Here α_j is the angle formed by the jth rectilinear segment of the boundary with the x-axis (in the direction of travel round the contour $L + M$, L is the unknown contour).

We transform to the parametric plane of the complex variable ζ by means of the transformation $z = \omega(\zeta)$. The analytic function $\omega(\zeta)$ conformally maps the exterior of two sections of the real axis in the plane of ζ into the doubly connected region in the plane of z occupied by the plate and bounded by the contour $L + M$. On the basis of Hilbert's theorem [8 and 9] and the result of Chaplygin [10] such a conformal transformation is always possible. In addition, it is evident that the contours L and M correspond to different sections. The images of the contours L and M in the ζ-plane will also be denoted by L and M .

In the ζ-plane we obtain from Formulas (3.2) the following boundary-value problem for the three analytic functions $\omega(\zeta)$, $\varphi(\zeta) = \Phi[\omega(\zeta)]$ and $\psi(\zeta) = \Psi[\omega(\zeta)]$:

$$\operatorname{Re}\varphi(\zeta) = 0 \quad \text{on } M, \qquad \operatorname{Re}\varphi(\zeta) = -A \quad \text{on } L \tag{3.3}$$

$$\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\varphi'(\zeta) + \psi(\zeta) = 0 \quad \text{on } L$$

$$\operatorname{Re}\left\{e^{2i\alpha_j}\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\varphi'(\zeta) + \psi(\zeta)\right]\right\} = 0 \quad \text{on } M, \quad \operatorname{Im}[e^{-i\alpha_j}\omega(\zeta)] = d_j \tag{3.4}$$

The last condition of (3.4) is an expression in complex form of the equation of the jth rectilinear segment of the boundary $y = x \tan \alpha_j + d_j \sec \alpha_j$.

We introduce the analytic function $\chi(\zeta)$

$$\chi(\zeta) = \frac{\omega'(\zeta)}{\varphi'(\zeta)} \psi(\zeta) \tag{3.5}$$

With the aid of this function we can write the boundary-value problem (3.4) in the form

$$\overline{\omega(\zeta)} + \chi(\zeta) = 0 \quad \text{on } L \tag{3.6}$$

$$\operatorname{Im}[e^{-i\alpha_j}\omega(\zeta)] = d_j, \qquad \operatorname{Im}[e^{i\alpha_j}\chi(\zeta)] = d_j \quad \text{on } M$$

By means of functions $\omega_1(\zeta)$ and $\chi_1(\zeta)$

$$\omega_1(\zeta) = \overline{\omega}(\zeta), \; \chi_1(\zeta) = \overline{\chi}(\zeta)$$

which are analytic outside the segments $L + M$ we can write the boundary-value problem (3.6) in the form of a Riemann boundary-value problem for the four functions $\omega(\zeta), \chi(\zeta), \omega_1(\zeta), \chi_1(\zeta)$

$$\begin{array}{ll} 1)\ \chi^+ + \omega_1^- = 0 \ \text{ on } L, & 5)\ \omega^+ - e^{2i\alpha_j}\omega_1^- = 2id_je^{i\alpha_j} \ \text{ on } M \\ 2)\ \omega^+ + \chi_1^- = 0 \ \text{ on } L, & 6)\ \omega^- - e^{2i\alpha_j}\omega_1^+ = 2id_je^{i\alpha_j} \ \text{ on } M \\ 3)\ \omega_1^+ + \chi^- = 0 \ \text{ on } L, & 7)\ \chi^+ - e^{-2i\alpha_j}\chi_1^- = 2id_je^{-i\alpha_j} \ \text{ on } M \\ 4)\ \omega^- + \chi_1^+ = 0 \ \text{ on } L, & 8)\ \chi^- - e^{-2i\alpha_j}\chi_1^+ = 2id_je^{-i\alpha_j} \ \text{ on } M \end{array} \tag{3.7}$$

We introduce new analytic functions

$$\varphi_1(\zeta) = \omega(\zeta) + \chi_1(\zeta), \qquad \varphi_2(\zeta) = \chi(\zeta) + \omega_1(\zeta) \tag{3.8}$$

Adding the first and third conditions of the boundary-value problem (3.7), also the second and fourth, and subtracting the eighth condition from the fifth and the seventh from the sixth, we obtain the following boundary-value problem for the functions $\varphi_1(\zeta)$ and $\varphi_2(\zeta)$

$$\begin{array}{ll} \varphi_2^+ + \varphi_2^- = 0, & \varphi_1^+ + \varphi_1^- = 0 \quad \text{on } L \\ \varphi_1^+ - e^{2i\alpha_j}\varphi_2^- = 0, & \varphi_2^+ - e^{-2i\alpha_j}\varphi_1^- = 0 \quad \text{on } M \end{array} \tag{3.9}$$

(3.9) is a particular case of the boundary-value problem considered in the preceding Section. Its solution can be found in closed form.

Substituting the functions $\omega_1(\zeta)$ and $\chi_1(\zeta)$ as defined by (3.8) and (3.9) into boundary-value problem (3.7), we obtain the following boundary-value problem of the Riemann type for determining the two functions $\omega(\zeta)$ and $\chi(\zeta)$:

$$\begin{array}{l} \chi^+ - \chi^- = \varphi_2^+, \qquad \omega^+ - \omega^- = \varphi_1^+ \quad \text{on } L \\ \omega^+ + e^{2i\alpha_j}\chi^- = 2id_je^{i\alpha_j} + \varphi_1^+ \quad \text{on } M \\ \chi^+ + e^{-2i\alpha_j}\omega^- = 2id_je^{-i\alpha_j} + \varphi_2^+ \quad \text{on } M \end{array} \tag{3.10}$$

Boundary-value problem (3.10) belongs to the class of Riemann boundary-value problems considered in the previous Section.

Thus the solution to boundary-value problem (3.6) and also the initial elastic problem can be found in quadratures. Note that the method of solution employed is applicable without any significant alterations to the case when the region occupied by the plate is triply connected.

ii) Consider now an absolutely rigid paraboloid of revolution pressing on a membrane of polygonal shape occupying a simply connected region. On the boundary M of the membrane we have the condition

$$w = 0 \tag{3.11}$$

We assume that the membrane is under uniform tension in all directions, i.e. that $\sigma_x = \sigma_y$ or $\varkappa = 1$.

By means of Formulas (1.9), (1.11) and (3.11) the boundary conditions of the problem can be written in the form

$$\operatorname{Re} f(z) = 0 \quad \text{on } M, \qquad f'(z) = -2A\bar{z} \quad \text{on } L \tag{3.12}$$

We transfer to the parametric plane of the complex variable by means of the function $z = \omega(\zeta)$. The analytic function $\omega(\zeta)$ transforms conformally the exterior of two sections of the real axis in the ζ-plane into the doubly-connected region in the z-plane occupied by the membrane and bounded by the contour $L + M$.

In the ζ-plane we obtain from Formulas (3.12) the following boundary-value problem for the two analytic functions $\omega(\zeta)$ and $F(\zeta) = f'[\omega(\zeta)]$

$$\begin{gathered} \operatorname{Re}\,[e^{i\alpha_j} F(\zeta)] = 0, \quad \operatorname{Im}\,[e^{-i\alpha_j} F(\zeta)] = d_j \quad \text{on } M \\ F(\zeta) = -2A\,\overline{\omega(\zeta)} \quad \text{on } L \end{gathered} \tag{3.13}$$

The second condition is the expression in complex form of the equation of the jth rectilinear segment of the boundary (as in Formula (3.4)).

With the aid of the functions

$$\omega_1(\zeta) = \overline{\omega}(\zeta), \qquad F_1(\zeta) = \overline{F}(\zeta)$$

which are analytic outside $L + M$ we can write the boundary-value problem (3.13) in the form of a Riemann boundary-value problem for the four functions $\omega(\zeta)$, $F(\zeta)$, $\omega_1(\zeta)$ and $F_1(\zeta)$

$$\begin{gathered} 1)\ F^+ + 2A\omega_1^- = 0, \quad 2)\ F^- + 2A\omega_1^+ = 0 \quad \text{on } L \\ 3)\ F_1^- + 2A\omega^+ = 0, \quad 4)\ F_1^+ + 2A\omega^- = 0 \quad \text{on } L \\ 5)\ F^+ + e^{-2i\alpha_j} F_1^- = 0, \quad 6)\ F^- + e^{-2i\alpha_j} F_1^+ = 0 \quad \text{on } M \\ 7)\ \omega^+ - e^{2i\alpha_j}\omega_1^- = 2id_j e^{i\alpha_j} \quad \text{on } M \\ 8)\ \omega^- - e^{2i\alpha_j}\omega_1^+ = 2id_j e^{i\alpha_j} \quad \text{on } M \end{gathered} \tag{3.14}$$

We introduce new analytic functions

$$\varphi_1(\zeta) = F(\zeta) + 2A\omega_1(\zeta), \qquad \varphi_2(\zeta) = F_1(\zeta) - 2A\omega(\zeta) \tag{3.15}$$

Adding the first and second conditions of the boundary-value problem

(3.14), subtracting the fourth from the third, multiplying the fifth and sixth conditions by $(-\frac{1}{2}A)$ and adding them, respectively, to the eighth and seventh, we obtain the following boundary-value problem for the functions $\varphi_1(\zeta)$ and $\varphi_2(\zeta)$

$$\begin{aligned} &\varphi_1^+ + \varphi_1^- = 0 \quad \text{on } L, \quad \varphi_1^+ + e^{-2i\alpha_j}\varphi_2^- = -4A\, id_j e^{-i\alpha_j} \quad \text{on } M \\ &\varphi_2^+ - \varphi_2^- = 0 \quad \text{on } L, \quad \varphi_2^+ + e^{2i\alpha_j}\varphi_1^- = -4A\, id_j e^{i\alpha_j} \quad \text{on } M \end{aligned} \tag{3.16}$$

(3.16) is a particular case of the boundary-value problem considered in the preceding Section. Its solution can be found in quadratures.

Substituting the functions $\omega_1(\zeta)$ and $F_1(\zeta)$ according to (3.15) and (3.16) into the boundary-value problem (3.14) we again arrive at the Riemann boundary-value problem for determining the two functions $\omega(\zeta)$ and $F(\zeta)$

The method of solution of the initial problem (3.12) can be applied without any significant change to the case of a triply connected region.

If the region occupied by the plate or membrane and situated outside the area of contact is simply-connected (or perhaps, is made simply-connected by the existence of lines of symmetry), then as a canonical region in the parametric plane of ζ it is more convenient to use a half-plane or a circle. In this case a solution of the boundary-value problems completely analogous to the already described is very much simplified since the functions $\omega_1(\zeta)$ and $F_1(\zeta)$ will be analytic in the region symmetrical to the region of definition of the functions $\omega(\zeta)$ and $F(\zeta)$ and not intersecting the latter, so that a pair of functions $\omega(\zeta)$ and $\omega_1(\zeta)$ (and also $F(\zeta)$ and $F_1(\zeta)$) can be considered as one piecewise-analytic function.

4. **Some specific problems.** (i) P r e s s u r e o f a p a r a b o l o i d o n a n i n f i n i t e m e m b r a n e . Consider an arbitrary rigid paraboloid, the equation of the surface of which is given by

$$z = Ax^2 + By^2 + Cxy \tag{4.1}$$

and which is bearing with a force P on an infinite arbitrarily tensioned membrane for which Expressions (1.9) are valid.

On the unknown boundary L of the contact area we have the boundary conditions

$$\operatorname{Re} f'(x) = -2Ax - Cy, \qquad \varkappa \operatorname{Im} f'(z) = 2By + Cx \quad \text{on } L \tag{4.2}$$
$$f'(z) \to 0 \qquad \text{for } z \to \infty$$

We transfer to the exterior of the unit circle in the parametric plane of ζ by means of the conformal transformation $z = \omega(\zeta)$. We thus obtain from (4.2) the following boundary-value problem for the two analytic functions $\omega(\zeta)$ and $F(\zeta) = f'[\omega(\zeta)]$:

$$(1+\varkappa)F(\zeta) + (1-\varkappa)\overline{F(\zeta)} = 2(B - A + Ci)\,\omega(\zeta) - 2(A+B)\overline{\omega(\zeta)} \quad \text{for } |\zeta| = 1$$
$$\omega(\zeta) = O(\zeta), \qquad F(\zeta) = O(\zeta^{-1}) \qquad \text{for } \zeta \to \infty \tag{4.3}$$

Either by using the method of functional equations [11], or by reducing (4.3) to a Riemann boundary-value problem, we can easily find a solution to the problem (4.3)

$$\omega(\zeta) = c_0\zeta + c_1/\zeta^{-1}, \qquad F(\zeta) = c_2/\zeta^{-1} \tag{4.4}$$

Here

$$|c_0^2| = \frac{P\,[4\,(B+\varkappa A)^2 + C^2\,(\varkappa-1)^2]}{8\pi\varkappa h\sigma_y\,(\varkappa^2 A + B)\,(4AB - C^2)}$$

$$\arg c_0 = \frac{1}{2}\left[\tan^{-1}\frac{(\varkappa-1)\,C}{2\,(B+\varkappa A)} - \tan^{-1}\frac{(\varkappa+1)\,C}{2\,(B-\varkappa A)}\right]$$

$$c_1 = \bar{c}_0\,\frac{2B - 2\varkappa A - C\,(\varkappa+1)\,i}{2B + 2\varkappa A - C\,(\varkappa-1)\,i}$$

$$c_2 = \frac{2\bar{c}_0}{\varkappa+1}\left[-A - B + (B - A + Ci)\,\frac{2B - 2\varkappa A - C\,(\varkappa+1)\,i}{2B + 2\varkappa A - C\,(\varkappa-1)\,i}\right]$$

The contact area is found to be elliptic.

In the solution use was also made of the condition of equilibrium of the paraboloid

$$P = qS, \qquad q = 2h\sigma_y\,(\varkappa^2 A + B) \tag{4.5}$$

where q is the pressure on the contact area and S is its area.

Note a particular case of the solution of (3.4) when $C = 0$, $\varkappa = 1$

$$c_0^2 = \frac{P\,(A+B)}{8\pi\sigma hAB} \qquad c_1 = \frac{B-A}{B+A}\,c_0, \qquad c_2 = -\frac{4AB}{A+B}\,c_0 \tag{4.6}$$

ii) Pressure of a paraboloid on an infinite plate. Consider a rigid paraboloid the equation of the surface of which is

$$z = Ax^2 + By^2 \tag{4.7}$$

pressing with a force P on an infinite plate for which the relations (1.3) are valid. This problem was first solved by a different method by Galin [1].

On the unknown boundary L of the contact area we obviously have the boundary conditions

$$2\,\mathrm{Re}\,\Phi(z) = -A - B, \quad \bar{z}\Phi'(z) + \Psi(z) = B - A \quad \text{on} \quad L \tag{4.8}$$

At a sufficient distance from the contact area the following relations hold [1]:

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= \frac{P}{4\pi D}\left(\ln\frac{x^2+y^2}{R^2} + 1\right) \\ \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - 2i\,\frac{\partial^2 w}{\partial x\,\partial y} &= \frac{P\,(x-iy)}{4\pi D\,(x+iy)} \end{aligned} \tag{4.9}$$

The asymptotic formulas (4.9) were derived on the assumption that the circular plate of radius R is fixed around the contour and that the dimensions of the contact area are small compared with the radius R.

We transfer to the exterior of the unit circle in the ζ-plane by means of the conformal transformation $z = \omega(\zeta)$. Then from (4.8) and (4.9) for the three analytic functions

$$\omega(\zeta), \quad \chi(\zeta) = \frac{\omega'(\zeta)}{\varphi'(\zeta)}\{B - A - \Psi[\omega(\zeta)]\}, \qquad \varphi(\zeta) = \Phi[\omega(\zeta)]$$

we can obtain the boundary-value problem

$$\text{for } |\zeta| = 1 \qquad \varphi(\zeta) + \overline{\varphi(\zeta)} = -A - B, \qquad \overline{\omega(\zeta)} = \chi(\zeta) \tag{4.10}$$

$$\text{for } \zeta \to \infty \qquad \omega(\zeta) = O(\zeta), \qquad c_0 = \omega'(\omega)$$

$$\varphi(\zeta) = \frac{P}{16\pi D}\left(\ln\frac{c_0^2\zeta^2}{R^2} + 1\right) + O(\zeta^{-1}) \qquad \chi(\zeta) = \frac{8\pi D}{P}\,c_0\,(B - A)\,\zeta + O(\zeta^{-1})$$

This problem can easily be solved; we find that

$$\varphi(\zeta) = \frac{P}{16\pi D}\left(2\ln\frac{c_0\zeta}{R} + 1\right), \qquad c_0 = R\exp\left[-\frac{4\pi D(A+B)}{P} - \frac{1}{2}\right]$$
$$\chi(\zeta) = c_0\left[\frac{8\pi D(B-A)}{P}\zeta + \frac{1}{\zeta}\right], \qquad \omega(\zeta) = c_0\left[\zeta + \frac{8\pi D(B-A)}{P\zeta}\right] \tag{4.11}$$

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